CHAPTER 3
FUZZY RELATION and COMPOSITION
**Definition (Product set)** Let $A$ and $B$ be two non-empty sets, the product set or Cartesian product $A \times B$ is defined as follows,

$$A \times B = \{(a, b) \mid a \in A, b \in B\}$$

**Example** $A = \{a_1, a_2, a_3\}$, $B = \{b_1, b_2\}$

$$A \times B = \{(a_1, b_1), (a_1, b_2), (a_2, b_1), (a_2, b_2), (a_3, b_1), (a_3, b_2)\}$$
Crisp relation

- **Binary Relation**
  If $A$ and $B$ are two sets and there is a specific property between elements $x$ of $A$ and $y$ of $B$, this property can be described using the ordered pair $(x, y)$. A set of such $(x, y)$ pairs, $x \in A$ and $y \in B$, is called a relation $R$.

  $$R = \{ (x, y) \mid x \in A, y \in B \}$$

- **$n$-ary relation**
  For sets $A_1, A_2, A_3, ..., A_n$, the relation among elements $x_1 \in A_1, x_2 \in A_2, x_3 \in A_3, ..., x_n \in A_n$ can be described by $n$-tuple $(x_1, x_2, ..., x_n)$. A collection of such $n$-tuples $(x_1, x_2, x_3, ..., x_n)$ is a relation $R$ among $A_1, A_2, A_3, ..., A_n$.

  $$(x_1, x_2, x_3, ..., x_n) \in R, \hspace{1cm} R \subseteq A_1 \times A_2 \times A_3 \times ... \times A_n$$
Crisp relation

- **Domain and Range**
  
  \[ dom(R) = \{ x \mid x \in A, (x, y) \in R \text{ for some } y \in B \} \]
  
  \[ ran(R) = \{ y \mid y \in B, (x, y) \in R \text{ for some } x \in A \} \]
Crisp relation

* Characteristics of relation
  
  (1) Surjection (many-to-one)
    
    - $f(A) = B$ or $\text{ran}(R) = B$. $\forall y \in B, \exists x \in A, y = f(x)$
    
    - Thus, even if $x_1 \neq x_2$, $f(x_1) = f(x_2)$ can hold.
(3) Injection (into, one-to-one)
- for all $x_1, x_2 \in A$, $x_1 \neq x_2$, $f(x_1) \neq f(x_2)$.
- if $R$ is an injection, $(x_1, y) \in R$ and $(x_2, y) \in R$ then $x_1 = x_2$.

(4) Bijection (one-to-one correspondence)
- both a surjection and an injection.
Crisp relation

## Representation methods of relations

1. Bipartigraph (Fig 3.7)  
   representing the relation by drawing arcs or edges

2. Coordinate diagram (Fig 3.8)  
   plotting members of $A$ on $x$ axis and that of $B$ on $y$ axis

Fig 3.7  Binary relation from $A$ to $B$  
Fig 3.8  Relation of $x^2 + y^2 = 4$
Crisp relation

(3) Matrix
manipulating relation matrix

(4) Digraph
the directed graph or digraph method

\[ M_R = (m_{ij}) \]

\[ m_{ij} = \begin{cases} 1, & (a_i, b_j) \in R \\ 0, & (a_i, b_j) \notin R \end{cases} \]

\[ i = 1, 2, 3, \ldots, m \]
\[ j = 1, 2, 3, \ldots, n \]

\[ R \]

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Matrix

Directed graph
Crisp relation

Operations on relations

\( R, S \subseteq A \times B \)

1. Union of relation \( T = R \cup S \)
   
   If \((x, y) \in R\) or \((x, y) \in S\), then \((x, y) \in T\)

2. Intersection of relation \( T = R \cap S \)
   
   If \((x, y) \in R\) and \((x, y) \in S\), then \((x, y) \in T\).

3. Complement of relation
   
   If \((x, y) \notin R\), then \((x, y) \in \overline{R}\)

4. Inverse relation
   
   \( R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R, x \in A, y \in B\} \)

5. Composition \( T \)
   
   \( R \subseteq A \times B, S \subseteq B \times C \), \( T = S \circ R \subseteq A \times C \)

   \( T = \{(x, z) \mid x \in A, y \in B, z \in C, (x, y) \in R, (y, z) \in S\} \)
Crisp relation

- **Path and connectivity in graph**
  - Path of length \( n \) in the graph defined by a relation \( R \subseteq A \times A \) is a finite series of \( p = a, x_1, x_2, \ldots, x_{n-1}, b \) where each element should be \( a \ R x_1, x_1 \ R x_2, \ldots, x_{n-1} \ R b \).
  - Besides, when \( n \) refers to a positive integer

(1) Relation \( R^n \) on \( A \) is defined, \( x \ R^n \ y \) means there exists a path from \( x \) to \( y \) whose length is \( n \).

(2) Relation \( R^\infty \) on \( A \) is defined, \( x \ R^\infty \ y \) means there exists a path from \( x \) to \( y \).

That is, there exists \( x \ R \ y \) or \( x \ R^2 \ y \) or \( x \ R^3 \ y \) ... and. This relation \( R^\infty \) is the reachability relation, and denoted as \( x R^\infty y \).

(3) The reachability relation \( R^\infty \) can be interpreted as connectivity relation of \( A \).
Properties of relation on a single set

**Fundamental properties**

1) Reflexive relation
\[ x \in A \rightarrow (x, x) \in R \text{ or } \mu_R(x, x) = 1, \ \forall \ x \in A \]
- irreflexive
  - if it is not satisfied for some \( x \in A \)
- antireflexive
  - if it is not satisfied for all \( x \in A \)

2) Symmetric relation
\[ (x, y) \in R \rightarrow (y, x) \in R \text{ or } \mu_R(x, y) = \mu_R(y, x), \ \forall \ x, y \in A \]
- asymmetric or nonsymmetric
  - when for some \( x, y \in A, (x, y) \in R \) and \( (y, x) \notin R \).
- antisymmetric
  - if for all \( x, y \in A, (x, y) \in R \) and \( (y, x) \notin R \)
3) Transitive relation
   For all $x, y, z \in A$
   $(x, y) \in R, (y, z) \in R \rightarrow (x, z) \in R$

4) Closure
   - Closure of $R$ with the respect to a specific property is the smallest relation $R'$ containing $R$ and satisfying the specific property
Properties of relation on a single set

Example

The transitive closure (or reachability relation) \( R^\infty \) of \( R \)
for \( A = \{1, 2, 3, 4\} \) and \( R = \{(1, 2), (2, 3), (3, 4), (2, 1)\} \) is
\[
R^\infty = R \cup R^2 \cup R^3 \cup \ldots
\]
\[
=\{(1, 1), (1, 2), (1, 3), (1,4), (2,1), (2,2), (2, 3), (2, 4), (3, 4)\}.
\]

Fig 3.10 Transitive closure
Properties of relation on a single set

- **Equivalence relation**

  1. Reflexive relation
     \[ x \in A \rightarrow (x, x) \in R \]
  2. Symmetric relation
     \[ (x, y) \in R \rightarrow (y, x) \in R \]
  3. Transitive relation
     \[ (x, y) \in R, (y, z) \in R \rightarrow (x, z) \in R \]
Properties of relation on a single set

- **Equivalence classes**

  A partition of $A$ into $n$ disjoint subsets $A_1, A_2, \ldots, A_n$

(a) Expression by set

(b) Expression by undirected graph

Fig 3.11  Partition by equivalence relation

$\pi(A/R) = \{A_1, A_2\} = \{\{a, b, c\}, \{d, e\}\}$
Properties of relation on a single set

- **Compatibility relation (tolerance relation)**
  - (1) Reflexive relation
    \[ x \in A \rightarrow (x, x) \in R \]
  - (2) Symmetric relation
    \[ (x, y) \in R \rightarrow (y, x) \in R \]

(a) Expression by set
(b) Expression by undirected graph

Fig 3.12 Partition by compatibility relation
Properties of relation on a single set

- **Pre-order relation**
  
  (1) **Reflexive relation**
  
  \[ x \in A \rightarrow (x, x) \in R \]

  (2) **Transitive relation**

  \[(x, y) \in R, (y, z) \in R \rightarrow (x, z) \in R\]
Properties of relation on a single set

♦ Order relation

(1) Reflexive relation
\[ x \in A \rightarrow (x, x) \in R \]

(2) Antisymmetric relation
\[ (x, y) \in R \rightarrow (y, x) \notin R \]

(3) Transitive relation
\[ (x, y) \in R, (y, z) \in R \rightarrow (x, z) \in R \]

strict order relation

(1’) Antireflexive relation
\[ x \in A \rightarrow (x, x) \notin R \]

total order or linear order relation

(4) \( \forall x, y \in A, (x, y) \in R \) or \( (y, x) \in R \)
## Properties of relation on a single set

### Comparison of relations

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<tr>
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Fuzzy relation

- **Crisp relation**
  - membership function $\mu_R(x, y)$
  
  \[
  \mu_R(x, y) = \begin{cases} 
  0 & \text{iff } (x, y) \notin R \\
  1 & \text{iff } (x, y) \in R 
  \end{cases}
  \]
  
  $\mu_R : A \times B \rightarrow \{0, 1\}$

- **Fuzzy relation**
  
  $\mu_R : A \times B \rightarrow [0, 1]$
  
  \[R = \{(x, y), \mu_R(x, y))| \mu_R(x, y) \geq 0, \ x \in A, \ y \in B\}\]
Example
Crisp relation $R$

$\mu_R(a, c) = 1$, $\mu_R(b, a) = 1$, $\mu_R(c, b) = 1$ and $\mu_R(c, d) = 1$.

Fuzzy relation $R$

$\mu_R(a, c) = 0.8$, $\mu_R(b, a) = 1.0$, $\mu_R(c, b) = 0.9$, $\mu_R(c, d) = 1.0$

(a) Crisp relation

(b) Fuzzy relation

Fig 3.16 crisp and fuzzy relations

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corresponding fuzzy matrix
Fuzzy relation

- **Fuzzy matrix**

  1) Sum
  \[
  A + B = \text{Max} \left[ a_{ij}, b_{ij} \right]
  \]

  2) Max product
  \[
  A \cdot B = AB = \text{Max} \left[ \text{Min} \left( a_{ik}, b_{kj} \right) \right]
  \]

  3) Scalar product
  \[
  \lambda A \text{ where } 0 \leq \lambda \leq 1
  \]
### Example

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Fuzzy relation

- **Operation of fuzzy relation**
  
  1) **Union relation**
     \[
     \forall (x, y) \in A \times B
     \]
     \[
     \mu_{R \cup S}(x, y) = \text{Max} [\mu_R(x, y), \mu_S(x, y)] = \mu_R(x, y) \lor \mu_S(x, y)
     \]
  
  2) **Intersection relation**
     \[
     \mu_{R \cap S}(x) = \text{Min} [\mu_R(x, y), \mu_S(x, y)] = \mu_R(x, y) \land \mu_S(x, y)
     \]
  
  3) **Complement relation**
     \[
     \forall (x, y) \in A \times B
     \]
     \[
     \mu_R(x, y) = 1 - \mu_R(x, y)
     \]
  
  4) **Inverse relation**
     
     For all \((x, y) \subseteq A \times B\),
     \[
     \mu_{R^{-1}}(y, x) = \mu_R(x, y)
     \]
Fuzzy relation

### Fuzzy relation matrix

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Fuzzy relation

Composition of fuzzy relation

For \((x, y) \in A \times B, (y, z) \in B \times C\),

\[
\mu_{SR}(x, z) = \max \left[ \min \left( \mu_R(x, y), \mu_S(y, z) \right) \right]
\]

\[
= \vee \left[ \mu_R(x, y) \land \mu_S(y, z) \right]
\]

\(M_{SR} = M_R \bullet M_S\)
Fuzzy relation

Example

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\[ S \circ R \]

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Composition of fuzzy relation
**Fuzzy relation**

- **α-cut of fuzzy relation**
  \[ R_\alpha = \{(x, y) \mid \mu_R(x, y) \geq \alpha, \ x \in A, y \in B\} : \text{a crisp relation.} \]

**Example**

\[ M_R = \begin{array}{ccc}
0.9 & 0.4 & 0.0 \\
0.2 & 1.0 & 0.4 \\
0.0 & 0.7 & 1.0 \\
0.4 & 0.2 & 0.0 \\
\end{array} \]

- \[ M_{R_{0.4}} = \begin{array}{ccc}
1 & 1 & 0 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
1 & 0 & 0 \\
\end{array} \]
- \[ M_{R_{0.7}} = \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 1 & 1 \\
0 & 0 & 0 \\
\end{array} \]
- \[ M_{R_{0.9}} = \begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{array} \]
- \[ M_{R_{1.0}} = \begin{array}{ccc}
0 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1 \\
0 & 0 & 0 \\
\end{array} \]
**Fuzzy relation**

- **Projection**

  For all \( x \in A, y \in B \),

  \[
  \mu_{R_A}(x) = \max_y \mu_R(x, y) \quad : \text{projection to A}
  \]

  \[
  \mu_{R_B}(y) = \max_x \mu_R(x, y) \quad : \text{projection to B}
  \]

- **Example**

  \[
  \begin{array}{c|ccc}
  & b_1 & b_2 & b_3 \\
  \hline
  a_1 & 0.1 & 0.2 & 1.0 \\
  a_2 & 0.6 & 0.8 & 0.0 \\
  a_3 & 0.0 & 1.0 & 0.3 \\
  \end{array}
  \]

  \[
  M_{R_A} = \begin{array}{c}
  a_1 \quad 1.0 \\
  a_2 \quad 0.8 \\
  a_3 \quad 1.0 \\
  \end{array}
  \]

  \[
  M_{R_B} = \begin{array}{ccc}
  & b_1 & b_2 & b_3 \\
  \hline
  0.6 & 1.0 & 1.0 \\
  \end{array}
  \]
Fuzzy relation

- Projection in $n$ dimension
  \[ \mu_{R_{X_{i1} \times X_{i2} \times \ldots \times X_{ik}}} (x_{i1}, x_{i2}, \ldots, x_{ik}) = \max_{x_{j1}, x_{j2}, \ldots, x_{jm}} \mu_R (x_1, x_2, \ldots, x_n) \]

- Cylindrical extension
  \[ \mu_{C(R)} (a, b, c) = \mu_R (a, b) \]
  \[ a \in A, b \in B, c \in C \]

- Example

  \[ \mu_{C(R_A)} (a_1, b_1) = \mu_{R_A} (a_1) = 1.0 \]
  \[ \mu_{C(R_A)} (a_1, b_2) = \mu_{R_A} (a_1) = 1.0 \]
  \[ \mu_{C(R_A)} (a_2, b_1) = \mu_{R_A} (a_2) = 0.8 \]

  \[ M_{C(R_A)} = \]
  \[
  \begin{array}{c|ccc}
  \text{a} & b_1 & b_2 & b_3 \\
  \hline
  a_1 & 1.0 & 1.0 & 1.0 \\
  a_2 & 0.8 & 0.8 & 0.8 \\
  a_3 & 1.0 & 1.0 & 1.0 \\
  \end{array}
  \]
Extension of fuzzy set

- **Extension by relation**
  - Extension of fuzzy set
    
    \[
    x \in A, \ y \in B \ \ y = f(x) \ \text{or} \ \ x = f^{-1}(y)
    \]
    
    for \( y \in B \)
    
    \[
    \mu_{B'}(y) = \max_{x \in f^{-1}(y)} \left[ \mu_A(x) \right] \quad \text{if} \ f^{-1}(y) \not= \emptyset
    \]

**Example**

\[A = \{(a_1, 0.4), (a_2, 0.5), (a_3, 0.9), (a_4, 0.6)\}, \ B = \{b_1, b_2, b_3\}\]

- \(f^{-1}(b_1) = \{(a_1, 0.4), (a_3, 0.9)\}\), \(\max [0.4, 0.9] = 0.9\)
  
  \[\Rightarrow \mu_{B'}(b_1) = 0.9\]

- \(f^{-1}(b_2) = \{(a_2, 0.5), (a_4, 0.6)\}\), \(\max [0.5, 0.6] = 0.6\)
  
  \[\Rightarrow \mu_{B'}(b_2) = 0.6\]

- \(f^{-1}(b_3) = \{(a_4, 0.6)\}\)
  
  \[\Rightarrow \mu_{B'}(b_3) = 0.6\]

\[B' = \{(b_1, 0.9), (b_2, 0.6), (b_3, 0.6)\}\]
Extension of fuzzy set

### Extension by fuzzy relation

For \( x \in A, y \in B, \) and \( B' \subseteq B \)

\[
\mu_{B'}(y) = \max \left[ \min \left( \mu_A(x), \mu_R(x, y) \right) \right]_{x \in f^{-1}(y)}
\]

#### Example

For \( b_1 \)

\[
\begin{align*}
\min [\mu_A(a_1), \mu_R(a_1, b_1)] &= \min [0.4, 0.8] = 0.4 \\
\min [\mu_A(a_3), \mu_R(a_3, b_1)] &= \min [0.9, 0.3] = 0.3 \\
\max [0.4, 0.3] &= 0.4 \quad \Rightarrow \quad \mu_{B'}(b_1) = 0.4
\end{align*}
\]

For \( b_2 \)

\[
\begin{align*}
\min [\mu_A(a_2), \mu_R(a_2, b_2)] &= \min [0.5, 0.2] = 0.2 \\
\min [\mu_A(a_4), \mu_R(a_4, b_2)] &= \min [0.6, 0.7] = 0.6 \\
\max [0.2, 0.6] &= 0.6 \quad \Rightarrow \quad \mu_{B'}(b_2) = 0.6
\end{align*}
\]

For \( b_3 \)

\[
\begin{align*}
\max \min [\mu_A(a_4), \mu_R(a_4, b_3)] &= \max \min [0.6, 0.4] = 0.4 \\
\Rightarrow \quad \mu_{B'}(b_3) = 0.4
\end{align*}
\]

\[ B' = \{(b_1, 0.4), (b_2, 0.6), (b_3, 0.4)\} \]
Example

\[ A = \{(a_1, 0.8), (a_2, 0.3)\} \]
\[ B = \{b_1, b_2, b_3\} \]
\[ C = \{c_1, c_2, c_3\} \]

\[ M_{R_1} = \]
\[
\begin{array}{c|ccc}
\text{} & b_1 & b_2 & b_3 \\
\hline
a_1 & 0.3 & 1.0 & 0.0 \\
a_2 & 0.8 & 0.0 & 0.0 \\
\end{array}
\]

\[ M_{R_2} = \]
\[
\begin{array}{c|ccc}
\text{} & c_1 & c_2 & c_3 \\
\hline
b_1 & 0.7 & 0.4 & 1.0 \\
b_2 & 0.2 & 0.0 & 0.8 \\
b_3 & 0.0 & 0.3 & 0.9 \\
\end{array}
\]

\[ B' = \{(b_1, 0.3), (b_2, 0.8), (b_3, 0)\} \]
\[ C' = \{(c_1, 0.3), (c_2, 0.3), (c_3, 0.8)\} \]
Fuzzy distance between fuzzy sets

- Pseudo-metric distance
  
  \[ \begin{align*}
  (1) \quad & d(x, x) = 0, \quad \forall \ x \in X \\
  (2) \quad & d(x_1, x_2) = d(x_2, x_1), \quad \forall \ x_1, x_2 \in X \\
  (3) \quad & d(x_1, x_3) \leq d(x_1, x_2) + d(x_2, x_3), \quad \forall \ x_1, x_2, x_3 \in X \\
  + \ (4) \ & \text{if } d(x_1, x_2) = 0, \text{ then } x_1 = x_2 \Rightarrow \text{ metric distance}
  \end{align*} \]

Distance between fuzzy sets

- \( \forall \ \delta \in \mathbb{R}^+, \mu_{d(A, B)}(\delta) = \text{Max} [\text{Min} (\mu_A(a), \mu_B(b))] \)
  
  \[ \delta = d(a, b) \]
Example: \[ A = \{(1, 0.5), (2, 1), (3, 0.3)\} \quad B = \{(2, 0.4), (3, 0.4), (4, 1)\} \]